Math 150, *Lecture Notes- Bonds* Name____________

Section 2.3 *Product and Quotient Rules and Higher-Order Derivatives*

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

PROOF

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] + \lim_{\Delta x\to 0}\left[g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0} f(x+\Delta x) \cdot \lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} + \lim_{\Delta x\to 0} g(x) \cdot \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Note:

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

Ex.1 Find the derivative of $g(x) = (-7x+9)(5x^3-4)$.

$g'(x) = (-7)(5x^3-4) + (15x^2)(-7x+9)$

$g'(x) = -35x^3+28 \;\; -105x^3 \;\; +135x^2$

$g'(x) = -140x^3 + 135x^2 + 28$

$\frac{d}{dx}[g(x)] = \frac{d}{dx}[(-7x+9)(5x^3-4)]$

$g'(x) = (-7x+9)\cdot\frac{d}{dx}[5x^3-4] + [5x^3-4]\cdot\frac{d}{dx}[-7x+9]$

$g'(x) = (-7x+9)[5\cdot\frac{d}{dx}(x^3) - \frac{d}{dx}(4)] + [5x^3-4]\cdot[-7\frac{d}{dx}(x) + \frac{d}{dx}(9)]$

$g'(x) = (-7x+9)[5\cdot 3x^2 - 0] + [5x^3-4]\cdot[-7\cdot 1 + 0]$

$g'(x) = (-7x+9)(15x^2) + (5x^3-4)(-7)$

$g'(x) = -105x^3 + 135x^2 - 35x^3 + 28$

$\boxed{g'(x) = -140x^3 + 135x^2 + 28}$

$x^5 + x^3 = x^3(x^2+1)$ $\quad t^{-1/2} \cdot t^{\frac{2}{2}} = t^{1/2}$ $\quad -\frac{1}{2}+\frac{2}{2}=\frac{1}{2}$

Ex.2 Find the derivative of $h(t)=\sqrt{t}\sin(t)$.

$\frac{d}{dt}[h(t)] = \frac{d}{dt}[t^{1/2} \cdot \sin(t)]$

$h'(t) = (t^{1/2}) \cdot \frac{d}{dt}[\sin(t)] + [\sin(t)] \cdot \frac{d}{dt}[t^{1/2}]$

$h'(t) = 1 \cdot t^{1/2} \cdot \cos(t) + [\sin(t)] \cdot [\frac{1}{2} t^{-1/2}]$

$h'(t) = \frac{1}{2} t^{-1/2} [2 \cdot t^{2/2} \cdot \cos(t) + \sin(t)]$

$h'(t) = \frac{1}{2} t^{-1/2} [2t\cos(t) + \sin(t)]$

THEOREM 2.8 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \qquad g(x) \neq 0$$

PROOF

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)}$$

$$= \frac{\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$ ■

$$[-5t+4]^2 = [-(5t-4)]^2 = (5t-4)^2$$

Ex.3 Find the derivative of $g(t)=\dfrac{3t^2-6}{-5t+4}$.

$$g'(t) = \frac{(-5t+4)(6t)-(3t^2-6)(-5)}{(-5t+4)^2}$$

$$g'(t) = \frac{-30t^2+24t+15t^2-30}{(-5t+4)^2}$$

$$g'(t) = \frac{-15t^2+24t-30}{(-5t+4)^2}$$

$$\boxed{g'(t) = \frac{-3(5t^2-8t+10)}{(5t-4)^2}}$$

Ex.4 Find the derivative of $f(a)=\dfrac{7\cos(a)}{6a^2}$.

$$f'(a) = \frac{(6a^2)\cdot\frac{d}{da}[7\cos(a)] - [7\cos(a)]\cdot\frac{d}{da}(6a^2)}{[6a^2]^2}$$

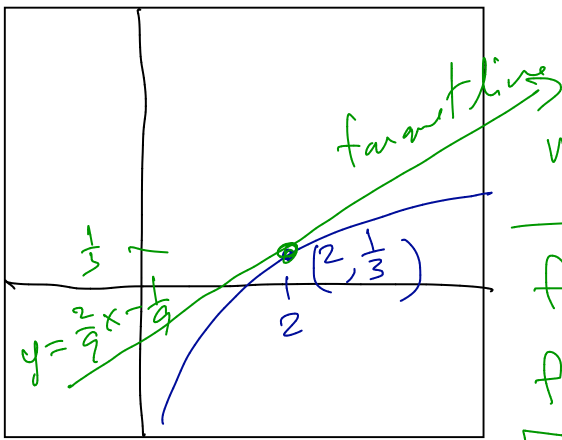
$$f'(a) = \frac{6a^2\cdot 7[-\sin(a)] - [7\cos(a)]\cdot[12a]}{36a^4}$$

$$f'(a) = \frac{-42a^2\sin(a) - 84a\cos(a)}{36a^4}$$

$$f'(a) = \frac{-6a[7a\sin(a)+14\cos(a)]}{6a\cdot[6a^3]}$$

$$f'(a) = \frac{-7[a\sin(a)+2\cos(a)]}{6a^3}$$

Ex.4 Find the equation of the tangent line to the graph of $f(x)=\dfrac{(x-1)}{(x+1)}$ at $\left(2,\dfrac{1}{3}\right)$. $=(x_1, y_1)$



$$y - y_1 = m_{tan}(x-x_1)$$

$$m_{tan} = \frac{2}{9}$$

$$f'(x) = \frac{(x+1)\cdot(1)-(x-1)(1)}{(x+1)^2}$$

$$f'(x) = \frac{x+1-x+1}{(x+1)^2}$$

$$\boxed{f'(x) = \frac{2}{(x+1)^2}}$$

$$m_{tan}\Big|_{(2,\frac{1}{3})} = f'(2) = \frac{2}{(2+1)^2} = \frac{2}{9}$$

$$y - \left(\tfrac{1}{3}\right) = \frac{2}{9}(x-2)$$

$$y - \frac{1}{3} = \frac{2}{9}x - \frac{4}{9}$$

$$y - \frac{1}{3} + \frac{1}{3} = \frac{2}{9}x - \frac{4}{9} + \frac{3}{9}$$

$$\boxed{y = \frac{2}{9}x - \frac{1}{9}}$$ tangent line at $(2,\frac{1}{3})$

Ex.5 Use of the Constant Multiple Rule:

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \dfrac{x^2 + 3x}{6}$	$y = \dfrac{1}{6}(x^2 + 3x)$	$y' = \dfrac{1}{6}(2x + 3)$	$y' = \dfrac{2x + 3}{6}$
b.	$y = \dfrac{5x^4}{8}$	$y = \dfrac{5}{8}x^4$	$y' = \dfrac{5}{8}(4x^3)$	$y' = \dfrac{5}{2}x^3$
c.	$y = \dfrac{-3(3x - 2x^2)}{7x}$	$y = -\dfrac{3}{7}(3 - 2x)$	$y' = -\dfrac{3}{7}(-2)$	$y' = \dfrac{6}{7}$
d.	$y = \dfrac{9}{5x^2}$	$y = \dfrac{9}{5}(x^{-2})$	$y' = \dfrac{9}{5}(-2x^{-3})$	$y' = -\dfrac{18}{5x^3}$

THEOREM 2.9 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Ex.6 Find the derivatives of the following functions:

(a) $\dfrac{d}{dx}\left[\tan(x)\right] = \dfrac{d}{dx}\left[\dfrac{\sin(x)}{\cos(x)}\right] = \dfrac{[\cos(x)]\cdot\frac{d}{dx}[\sin(x)] - [\sin(x)]\cdot\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$

$= \dfrac{[\cos(x)]\cdot[\cos(x)] - [\sin(x)]\cdot[-\sin(x)]}{[\cos(x)]^2}$

$= \dfrac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$

$= \dfrac{1}{\cos^2(x)} = \sec^2(x)$

(b) $\dfrac{d}{dx}\left[\sec(x)\right] = \dfrac{d}{dx}\left[\dfrac{1}{\cos(x)}\right]$

$= \dfrac{[\cos(x)]\cdot\frac{d}{dx}(1) - (1)\cdot\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$

$= \dfrac{[\cos(x)]\cdot 0 - 1\cdot(-\sin(x))}{\cos^2(x)}$

$= \dfrac{\sin(x)}{\cos^2(x)}$

$= \dfrac{1}{\cos(x)}\cdot\dfrac{\sin(x)}{\cos(x)}$

$= \sec(x)\tan(x)$

(c) $\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$

(d) $\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$

Ex.7 Find the derivative of $f(w) = \tan(w)\cot(w)$.

$$\frac{d}{dw}[f(w)] = \frac{d}{dw}\left[\frac{\sin(w)}{\cos(w)} \cdot \frac{\cos(w)}{\sin(w)}\right]$$

$$f'(w) = \frac{d}{dw}(1)$$

$$f'(w) = 0$$

Ex.8 Find the derivative of $h(\theta) = 5\theta\sec(\theta) + \theta\tan(\theta)$.

$$h'(\theta) = \frac{d}{d\theta}\left[5\theta \cdot \sec(\theta) + \theta \cdot \tan(\theta)\right]$$

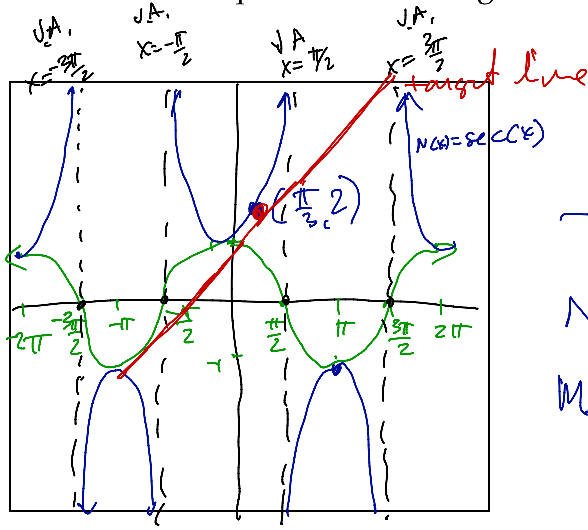
$$h'(\theta) = 5 \cdot \frac{d}{d\theta}\left[\theta \cdot \sec(\theta)\right] + \frac{d}{d\theta}\left[\theta \cdot \tan(\theta)\right]$$

$$h'(\theta) = 5 \cdot \left\{(\theta) \cdot \frac{d}{d\theta}\left[\sec(\theta)\right] + \left[\sec(\theta)\right] \cdot \frac{d}{d\theta}(\theta)\right] + \left\{(\theta) \cdot \frac{d}{d\theta}\left[\tan(\theta)\right] + \left[\tan(\theta)\right] \cdot \frac{d}{d\theta}(\theta)\right]$$

$$h'(\theta) = 5\left[\theta \cdot \sec(\theta)\tan(\theta) + \sec(\theta) \cdot 1\right] + \left[\theta \cdot \sec^2(\theta) + \tan(\theta) \cdot 1\right]$$

$$h'(\theta) = 5\theta\sec(\theta)\tan(\theta) + 5\sec(\theta) + \theta\sec^2(\theta) + \tan(\theta)$$

Ex.9 Find the equation of the tangent line to the graph of $N(x)=\sec(x)$ at $\left(\frac{\pi}{3},2\right)$. $=(x_1,y_1)$



$$y-y_1=m_{tan}(x-x_1)$$

$m_{tan}=$ ______

$$N'(x)=\sec(x)\tan(x)$$

$$m_{tan}\Big|_{(\frac{\pi}{3},2)}=N'\left(\frac{\pi}{3}\right)=\sec\left(\frac{\pi}{3}\right)\cdot\tan\left(\frac{\pi}{3}\right)$$

$$=\frac{1}{\cos\left(\frac{\pi}{3}\right)}\cdot\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$$

$$=\frac{1}{\frac{1}{2}}\cdot\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$\boxed{m_{tan}=2\cdot\sqrt{3}}$$

$$y-(2)=2\sqrt{3}\left[x-\left(\frac{\pi}{3}\right)\right]$$

$$y-2=2\sqrt{3}x-\frac{2\pi\sqrt{3}}{3}$$

$$y-2+2=2\sqrt{3}x-\frac{2\pi\sqrt{3}}{3}+\frac{6}{3}$$

$$\boxed{y=2\sqrt{3}x+\frac{6-2\pi\sqrt{3}}{3}}$$

Higher-Order Derivatives

NOTE The second derivative of f is the derivative of the first derivative of f.

Ex.10 Find the second derivative of $f(x)=8x^6-10x^5+5x^3$.

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[8x^6-10x^5+5x^3]$$

$$f'(x) = 8\cdot\frac{d}{dx}[x^6] - 10\cdot\frac{d}{dx}[x^5] + 5\cdot\frac{d}{dx}[x^3]$$

$$f'(x) = 8\cdot[6x^5] - 10[5x^4] + 5[3x^2]$$

$$f'(x) = 48x^5 - 50x^4 + 15x^2$$

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}[48x^5-50x^4+15x^2]$$

$$f''(x) = 48\cdot\frac{d}{dx}[x^5] - 50\frac{d}{dx}[x^4] + 15\frac{d}{dx}[x^2]$$

$$f''(x) = 48\cdot[5x^4] - 50\cdot[4x^3] + 15\cdot[2x]$$

$$f''(x) = 240x^4 - 200x^3 + 30x$$

$$f(x) = 8x^6-10x^5+5x^3$$

$$f'(x) = 48x^5 - 50x^4 + 15x^2$$

$$f''(x) = 240x^4 - 200x^3 + 30x$$

Ex.11 Find the third derivative of $f(x)=2-\frac{2}{x}$.

$$f(x) = 2-2x^{-1}$$

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[2] - 2\frac{d}{dx}[x^{-1}]$$

$$f'(x) = 0 - 2[-x^{-2}]$$

$$f'(x) = 2x^{-2}$$

$$\frac{d}{dx}[f'(x)] = 2\cdot\frac{d}{dx}[x^{-2}]$$

$$f''(x) = 2\cdot[-2x^{-3}]$$

$$f''(x) = -4x^{-3}$$

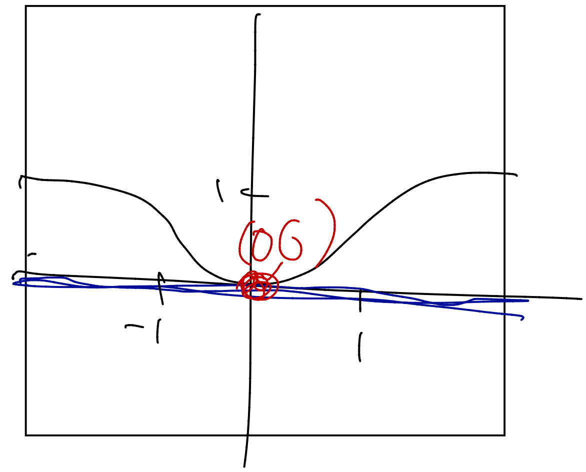
$$\frac{d}{dx}[f''(x)] = -4\cdot\frac{d}{dx}[x^{-3}]$$

$$f'''(x) = -4\cdot[-3x^{-4}]$$

$$f'''(x) = 12x^{-4}$$

$$f'''(x) = \frac{12}{x^4}$$

Ex.12 Determine the point(s) at which the graph of $g(x)=\dfrac{x^2}{x^2+1}$ has a horizontal tangent line.



$$\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[\frac{x^2}{x^2+1}\right]$$

$$g'(x) = \frac{(x^2+1)\frac{d}{dx}[x^2] - (x^2)\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$

$$g'(x) = \frac{(x^2+1)[2x] - (x^2)[2x]}{(x^2+1)^2}$$

$$g'(x) = \frac{\cancel{2x^3}+2x\cancel{-2x^3}}{(x^2+1)^2}$$

$$g'(x) = \frac{2x}{(x^2+1)^2}$$

$g'(x)$ tells us the slope of tangent line (m_{tan}) at a point on a curve.

Horizontal tangent line has slope of zero. ($m_{tan}=0$)

$g'(x)=0$

$\dfrac{2x}{(x^2+1)^2}=0$

$2x=0$

$x=0$

$g(x) = \dfrac{x^2}{x^2+1}$

$g(0) = \dfrac{(0)^2}{0^2+1}$

$g(0)=0$

$\boxed{(0,0)}$

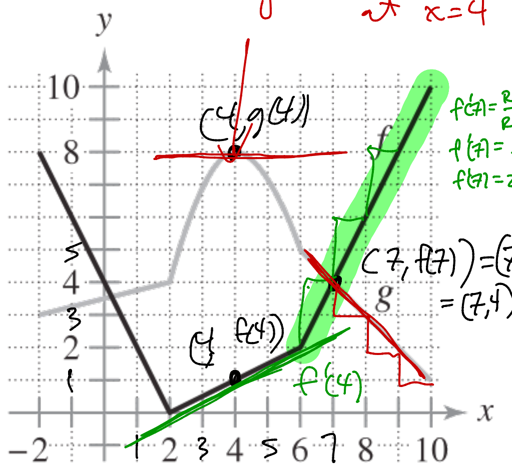
Ex.13 Given $p(x) = f(x)g(x)$ and $q(x) = \dfrac{f(x)}{g(x)}$, use the graphs of f and g to find the following derivatives:

(a) Find $p'(4)$.

(b) Find $q'(7)$.

$g'(4) = 0$

$g'(4) =$ slope of the tangent line at $x = 4$



$f'(7) = \dfrac{\text{Rise}}{\text{Run}}$

$f'(7) = \dfrac{2}{1}$

$f'(7) = 2$

(a)

$p(x) = f(x) \cdot g(x)$

$p'(x) = \dfrac{d}{dx}[f(x) \cdot g(x)]$

$p'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

$p'(4) = f(4) \cdot g'(4) + g(4) \cdot f'(4)$

$p'(4) = (1)(0) + (8)\left(\dfrac{1}{2}\right)$

$p'(4) = 4$

$(4, f(4)) = (4, 1)$

$(4, g(4)) = (4, 8)$

$g'(7) = \dfrac{\text{Rise}}{\text{Run}}$

$g'(7) = \dfrac{-1}{1}$

$g'(7) = -1$

$f'(4)$ is the slope of the tangent line at $x = 4$

$f'(4) = \dfrac{\text{Rise}}{\text{Run}}$

$f'(4) = \dfrac{1}{2}$

$$q'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

$$q'(7) = \frac{g(7) \cdot f'(7) - f(7)\, g'(7)}{[g(7)]^2}$$

$$q'(7) = \frac{(4)(2) - (4)(-1)}{(4)^2}$$

$$q'(7) = \frac{8 + 4}{16}$$

$$q'(7) = \frac{12}{16}$$

$$q'(7) = \frac{3}{4}$$

Ex.13 Given $p(x) = f(x)g(x)$ and $p(x) = \dfrac{f(x)}{g(x)}$, use the graphs of f and g to find the following derivatives:

(a) Find $p'(4)$.

(b) Find $q'(7)$.